Spinning black hole binaries for ET: SNR estimates and parameter estimation calculations

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Outline

- Motivation to study spinning black hole binaries
- Construction of gravitational waveform model: inspiral, transition, plunge and ring–down
- Fisher Matrix Analysis for a 3 ET detector network
- Results
- Conclusions and future work

Motivation

- Generalize our previous analysis for non–spinning BH binaries presented in Sicily
- "Static model": inspiral phase: "kludge–numerical model", merger phase: "EOB model", ring–down evolution
- Spinning black hole binaries are richer in information than their static counterparts
- Inspiral evolution of a compact object (CO) onto a spinning IMBH lasts longer and probes regions much closer to the light ring as compared with a static IMBH
- CO is subject to stronger relativistic effects at the end of inspiral evolution
- We can store more information in the Fisher Matrix
- Extend statistical analysis to study a 10D parameter space — 4 intrinsic parameters and 6 extrinsic ones
- Find out whether we can further improve extrinsic parameter determination using a detector network of 3 ETs

Gravitational waveform model

- Inspiral evolution for circular equatorial orbits is modelled using the "kludge waveform model" by Huerta & Gair (PhysRevD.79.084021)
- The basic ingredients are

$$\begin{aligned} \frac{\mathrm{d}\phi}{\mathrm{d}t} &\equiv \Omega = \frac{\sqrt{M}}{p^{3/2} \pm a\sqrt{M}} \\ \dot{p} &= \frac{\mathrm{d}p}{\mathrm{d}L_z}\dot{L}_z \end{aligned} \tag{1}$$

- The angular momentum flux $\dot{L}_z$ is tuned to mimic Teukolsky–based waveforms

$$\dot{L}_z = -\frac{32}{5}\frac{\mu^2}{M}\left(\frac{M}{p}\right)^{7/2}\Bigg\{1 - \frac{61}{12}q\left(\frac{M}{p}\right)^{3/2} - \frac{1247}{336}\left(\frac{M}{p}\right) + 4\pi\left(\frac{M}{p}\right)^{3/2} - \frac{44711}{9072}\left(\frac{M}{p}\right)^2 + \frac{33}{16}q^2\left(\frac{M}{p}\right)^2 + \text{high order Teukolsky fits}\Bigg\}. \quad (2)$$

- Overlap between this "numerical kludge" and Teukolsky–based waveforms is greater than 0.95 over a considerable portion of the parameter space
- This scheme breaks down slightly before the ISCO at a point $\tilde{r}_{\rm trans} = r_{\rm trans}/M$
- From this point onwards the orbit gradually changes from inspiral to plunge: "transition regime", cf. Ori & Thorne (PhysRevD.62.124022)
- Radiation reaction still drives the orbital evolution during the transition regime
- Because the CO moves on a circular orbit with radius very close to $\tilde{r}_{\rm trans}$ and its radiation reaction is weak, the equations of motion are given by

$$\frac{d\phi}{d\tilde{t}} \equiv \tilde{\Omega} \simeq \frac{1}{\tilde{r}_{\rm trans}^{3/2} + q} , \quad (3)$$

$$\frac{d\tilde{\tau}}{d\tilde{t}} \simeq \left(\frac{d\tilde{\tau}}{d\tilde{t}}\right)_{\rm trans} = \frac{\sqrt{1 - 3/\tilde{r}_{\rm trans} + 2q/\tilde{r}_{\rm trans}^{3/2}}}{1 + q/\tilde{r}_{\rm trans}^{3/2}} . \quad (4)$$

$$\frac{d^2 R}{d\tilde{\tau}^2} = -\alpha R^2 - \eta\beta\kappa\tilde{\tau} , \quad (5)$$

- where the various dimensionless quantities quoted above are given by

$$\frac{d\xi}{d\tilde{\tau}} = -\kappa\eta , \quad \text{and} \quad (6)$$

$$\kappa = \frac{32}{5}\tilde{\Omega}_{\rm trans}^{7/3} \frac{1 + q/\tilde{r}_{\rm trans}^{3/2}}{\sqrt{1 - 3/\tilde{r}_{\rm trans} + 2q/\tilde{r}_{\rm trans}^{3/2}}} \dot{\mathcal{E}}_{\rm trans} , \quad (7)$$

- $R \equiv \tilde{r} - \tilde{r}_{\rm trans}$ and $\xi \equiv \tilde{L} - \tilde{L}_{\rm trans}$ are introduced to Taylor expand Kerr's effective potential around $\tilde{r}_{\rm trans}$ and study the CO's location throughout the transition regime
- The constants α and β are functions of the Kerr effective potential evaluated at $\tilde{r}_{\rm trans}$, cf. Ori & Thorne (PhysRevD.62.124022)
- At some point the transition regime breaks down, radiation reaction becomes unimportant and pure plunge takes over with nearly constant orbital energy

and orbital angular momentum

$$\begin{aligned}\tilde{L}_{\rm fin} - \tilde{L}_{\rm trans} &= -(\kappa\tau_0 T_{\rm plunge})\eta^{4/5} \ , \\ \tilde{E}_{\rm fin} - \tilde{E}_{\rm trans} &= -\tilde{\Omega}_{\rm trans}(\kappa\tau_0 T_{\rm plunge})\eta^{4/5} \ , \end{aligned} \tag{8}$$

where,

$$T_{\rm plunge} = 3.412 \ , \qquad \tau_o = (\alpha\beta\kappa)^{-1/5} \ . \tag{9}$$

We now must replace the transition regime by the exact Kerr's metric adiabatic inspiral formulae

$$\frac{d^2\tilde{r}}{d\tilde{\tau}^2} = \frac{6\,\tilde{E}_{\rm fin}\,\tilde{L}_{\rm fin}\,q + \tilde{L}^2_{\rm fin}\,(\tilde{r}-3) + (q^2-\tilde{r})\tilde{r} - \tilde{E}^2_{\rm fin}\,q^2(\tilde{r}+3)}{\tilde{r}^4}, \tag{10}$$

$$\frac{d\phi}{d\tilde{t}} = \frac{\tilde{L}_{\rm fin}\,(\tilde{r}-2) + 2\,\tilde{E}_{\rm fin}\,q}{\tilde{E}_{\rm fin}\,(\tilde{r}^3 + (2+\tilde{r})\,q^2) - 2\,q\,\tilde{L}_{\rm fin}}, \tag{11}$$

$$\frac{d\tilde{\tau}}{d\tilde{t}} = \frac{\tilde{r}\,(q^2 + \tilde{r}\,(\tilde{r}-2))}{\tilde{E}_{\rm fin}\,(\tilde{r}^3 + (2+\tilde{r})\,q^2) - 2\,q\,\tilde{L}_{\rm fin}}. \tag{12}$$

- Match the transition regime onto the plunge phase at the point $\tilde{r}_{\rm plunge}$ where the transition angular frequency (3) and the plunge angular frequency (11) smoothly match for these specific values of energy and angular momentum (8).
- Up to now waveform model is well approximated using a flat–space–time wave emission formula, namely,

$$h(t) = -(h_+ - ih_\times) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h^{lm} \, {}_{-2}Y_{lm}(\theta, \Phi), \tag{13}$$

- ${}_{-2}Y_{lm}(\theta, \Phi)$ are the spin–weight -2 spherical harmonics. We shall consider only the modes $(l, m) = (2, \pm 2)$
- The RD waveform we shall build now originates from the distorted Kerr black hole formed after merger
- It is a superposition of quasinormal modes (l, m, n)
- Each mode has a complex frequency ω: real part is the oscillation frequency, imaginary part is the inverse of the damping time,

$$\omega = \omega_{lmn} - i/\tau_{lmn}. \tag{14}$$

- These two quantities are uniquely determined by the mass and angular momentum of the newly formed Kerr black hole
- Recent numerical studies (Berti & Cardoso, PhysRevD.76.064034) have shown that the energy released from inspiral to ringdown by maximally spinning BH binaries whose mass ratios are smaller than 1/10 ranges from 0.6% (antialigned configuration) – 1.5% (aligned configuration) of M and scales as η^2
- Hence, the one–fit function for the final mass of a distorted Kerr BH after merger derived by Buonanno et. al., (PhysRevD.76.044003) within the

framework of the EOB model should still provide a reasonable estimate (1.6%–1.8% of M) for spinning IMRIs

❄ The value of the final spin of the distorted Kerr black hole is obtained using the fit by Rezzolla, et. al., (ApJL, 2008)

$$a_f/M_f = q_f = q + s_4\, q^2\, \eta + s_5\, q\, \eta^2 + t_0\, q\, \eta + 2\sqrt{3}\, \eta + t_2\, \eta^2 + t_3\, \eta^3, \qquad (15)$$

a least–squares fit to available data yields,

$$\begin{aligned} s_4 = -0.129 &\pm 0.012, \qquad s_5 = 0.384 \pm 0.261, \\ t_0 = -2.686 &\pm 0.065, \qquad t_2 = 3.454 \pm 0.132, \\ t_3 &= 2.353 \pm 0.548. \end{aligned} \qquad (16)$$

❄ These fits allow us to compute the quasinormal frequencies (14) that describe the perturbations of a Kerr black hole during the RD phase

❄ These perturbations are usually described in terms of spin–weight -2 spheroidal harmonics $S_{lmn} = S_{lm}(a\omega_{lmn})$,

❄ Our ring–down waveform includes the fundamental mode ($l = 2, m = 2, n = 0$) and two overtones ($n = 1, 2$) and their respective "twin" modes with frequency $\omega'_{lmn} = -\omega_{l-mn}$ and a different damping $\tau' = \tau_{l-mn}$, i.e., (Berti, et. al., PhysRevD.73.064030)

$$h(t) = \frac{M}{D}\sum_{lmn}\Big\{\mathcal{A}_{lmn}e^{-i(\omega_{lmn}t+\phi_{lmn})}e^{-t/\tau_{lmn}}S_{lm}(a\omega_{lmn}) + \mathcal{A}'_{lmn}e^{i(\omega_{lmn}t+\phi'_{lmn})}e^{-t/\tau_{lmn}}S^{*}_{lm}(a\omega_{lmn})\Big\}. \quad (17)$$

- D is the distance to the source. Expanding ${}_{-2}S_{lm}^{a\omega_{\rm triad}}$ at first order will suffice for the analysis we shall carry out later on

$$ {}_{-2}S_{lm}^{a\omega_{\rm triad}} = {}_{-2}Y_{lm} + a\omega_{\rm triad}S^{(1)}_{lm} + (a\omega)^2, \quad (18)$$

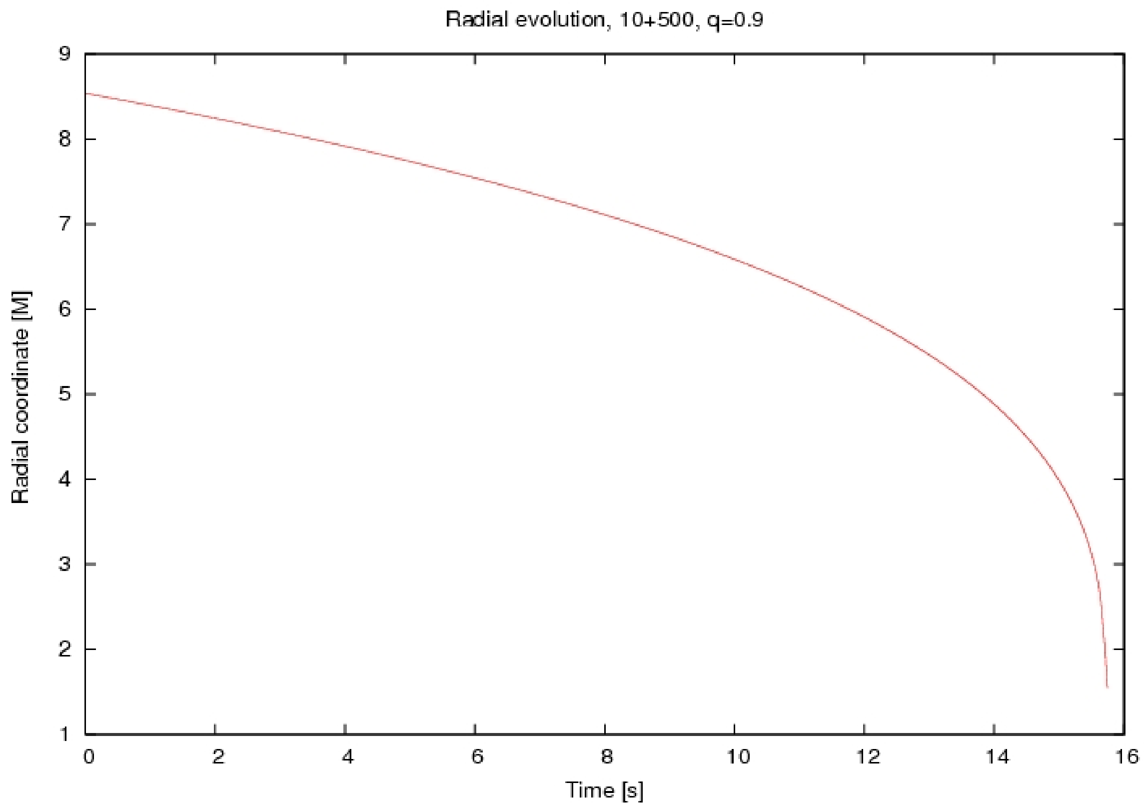
$$ S^{(1)}_{lm} = \sum_{l'} c^{l'}_{lm}\ {}_{-2}Y_{l'm}\,. \quad (19)$$

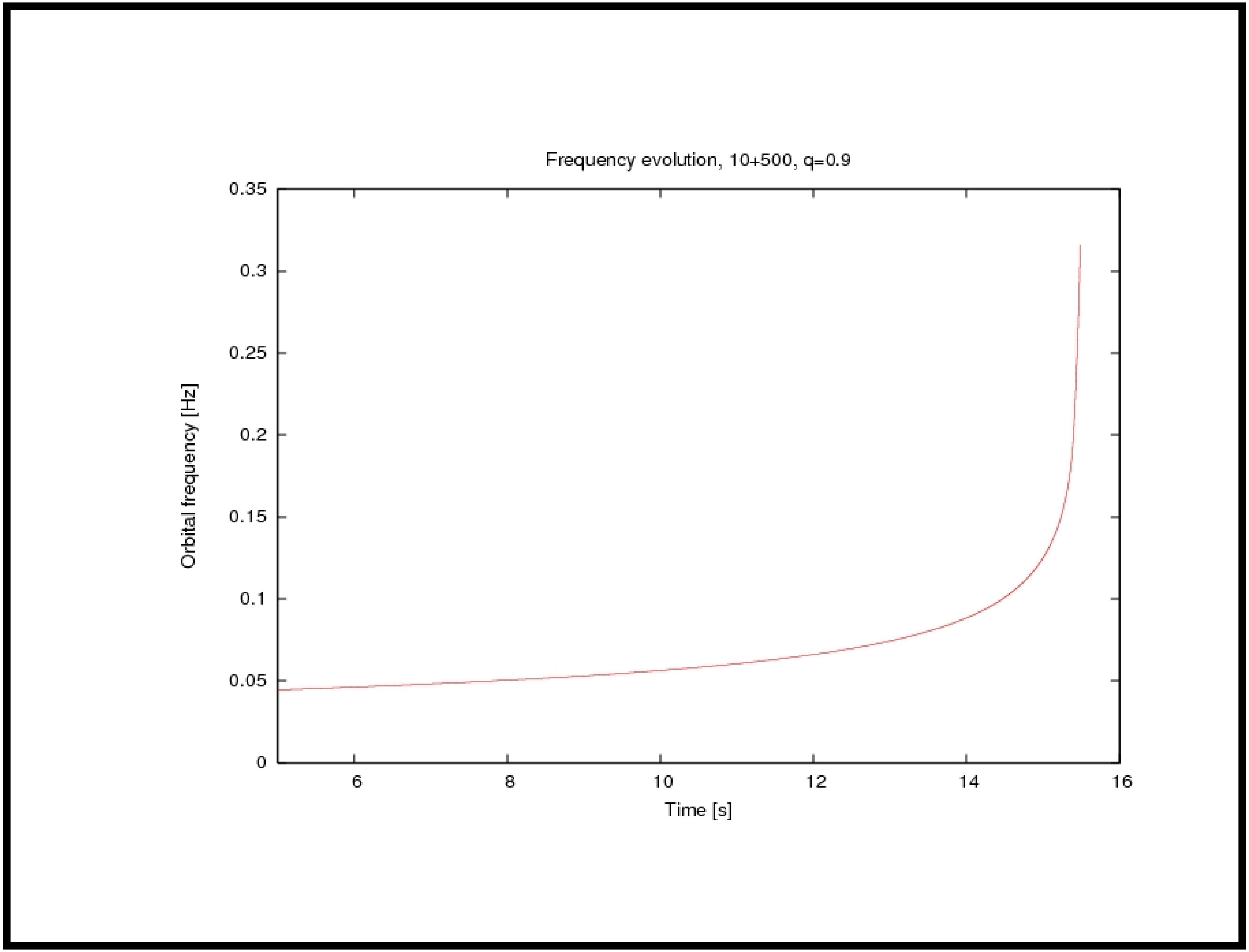
- Recall $S_{lmn} = S_{lm}(a\omega_{lmn})$, so $\omega_{\rm triad}$ is determined by the triad (l, m, n)
- The coefficients $c^{l'}_{lm}$ are computed using the relation

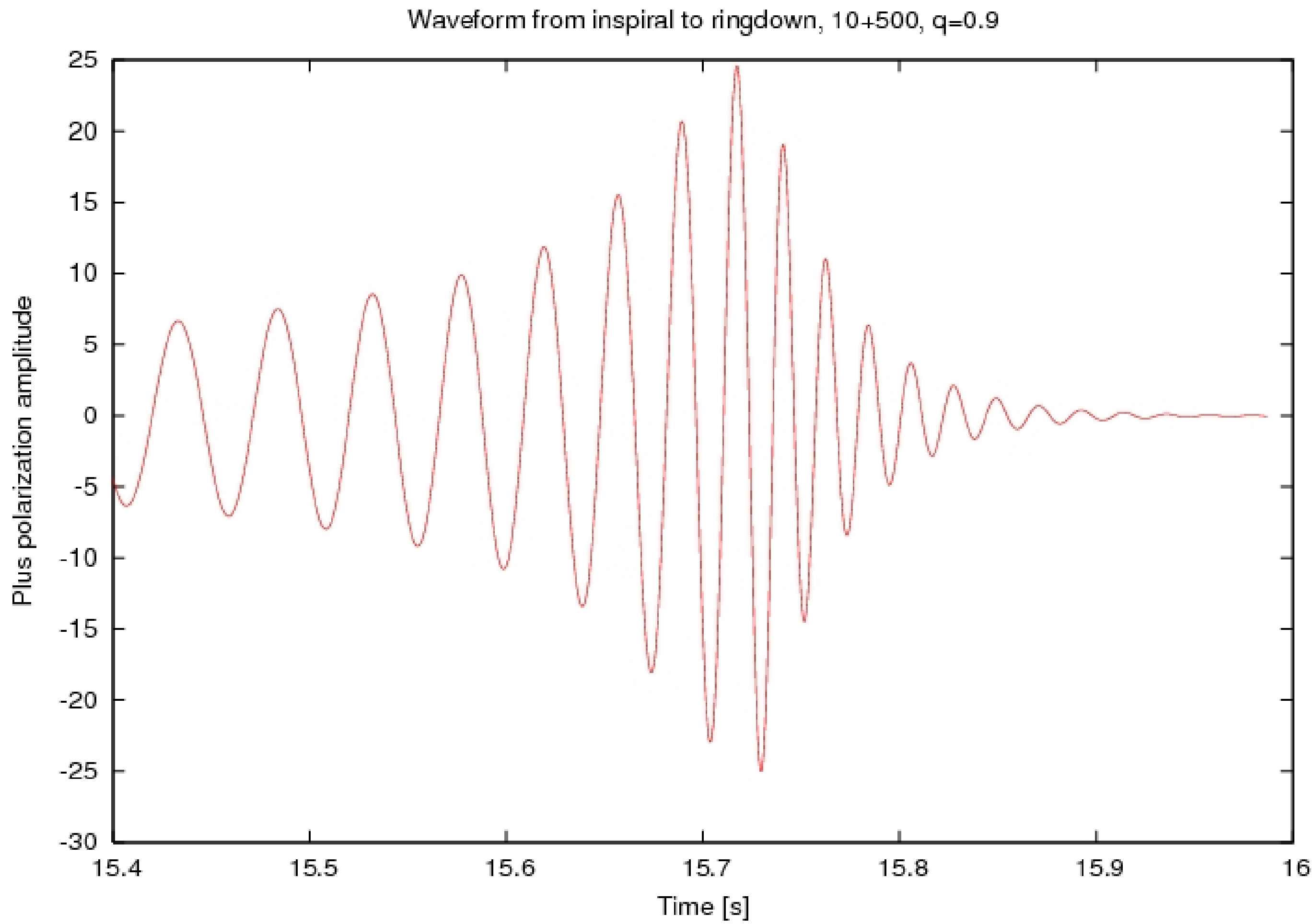
$$c^{l'}_{lm} = \begin{cases} \frac{4}{(l'-1)(l'+2)-(l-1)(l+2)}\int d(\cos\theta)\ {}_{-2}Y_{l'm}\cos\theta\ {}_{-2}Y_{lm}\,. & l' \neq l, \\ 0, & l' = l. \end{cases}$$

- Use these expressions to match the plus and cross RD polarizations onto their plunge counterparts

- This amounts to determine 24 constants, 12 for each polarization
- Use the plunge waveform to compute ten points before and after the RD to build an interpolation function: this solution is valid all the way to the horizon!
- Match onto the various quasinormal modes by imposing the continuity of the plunge and ringdown waveforms and all the necessary higher order time derivatives
- Match the plunge waveform onto the RD one using only the leading tone $n = 0$ at the time $t_{\rm peak}$ when the orbital frequency (11) peaks $\rightarrow$ fix 4 constants, 2 per polarization.
- Use these values as seed to compute the amplitudes and phases of the first overtone at $t_{\rm peak} + dt$
- Finally, use the values of the amplitudes and phases of the leading tone and first overtone to determine the four remaining constants at $t_{\rm peak} + 2dt$.
- The actual orbital and frequency evolution for a 10+500 $M_\odot$ binary system with q=0.9 along with its respective waveform from inspiral to ringdown looks as follows


Radial evolution, 10+500, q=0.9
Radial coordinate [M]
Time [s]
9
8
7
6
5
4
3
2
1
0
2
4
6
8
10
12
14
16


Frequency evolution, 10+500, q=0.9
Orbital frequency [Hz]
Time [s]
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
6
8
10
12
14
16


Waveform from inspiral to ringdown, 10+500, q=0.9
Plus polarization amplitude
Time [s]
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
15.4
15.5
15.6
15.7
15.8
15.9
16

FISHER MATRIX ANALYSIS

- Consider a detector network of three ETs in triangular configuration
- We will use the target "ET B" noise curve $S_n(f)$
- When computing the FMs for the various interferometers take into account the rotation of the Earth: initial radius of inspiral and initial phase of inspiral will be different for every detector
- Use the appropriate response function for a ground–based interferometer
- To compute the expectation value of the noise–induced errors we use the relation

$$\left\langle \Delta\theta^i \Delta\theta^j \right\rangle = (\Gamma^{-1})^{ij} + \mathcal{O}(\mathrm{SNR})^{-1}. \tag{20}$$

- FM is given by

$$\Gamma_{ab} = 2\sum_{\alpha} \int_0^T \partial_a \hat{h}_\alpha(t) \partial_b \hat{h}_\alpha(t) \mathrm{d}t\,, \tag{21}$$

$$\hat{h}_\alpha(t) \equiv \frac{h_\alpha(t)}{\sqrt{S_h\left(f(t)\right)}}, \qquad f(t) = \frac{1}{\pi}\frac{\mathrm{d}\phi}{\mathrm{d}t}. \tag{22}$$

- IMRI space is a 10D parameter space of signals: 4 intrinsic parameters and 6 intrinsic ones
- Complete waveforms last from seconds to a few minutes

Results

- ☆ We run MC for 4 different binary systems, namely, $(10{+}500)M_\odot$, $(1.4{+}500)$ $M_\odot$ with q=0.9 and q=0
- ☆ We summarize the results for two of them in the following tables

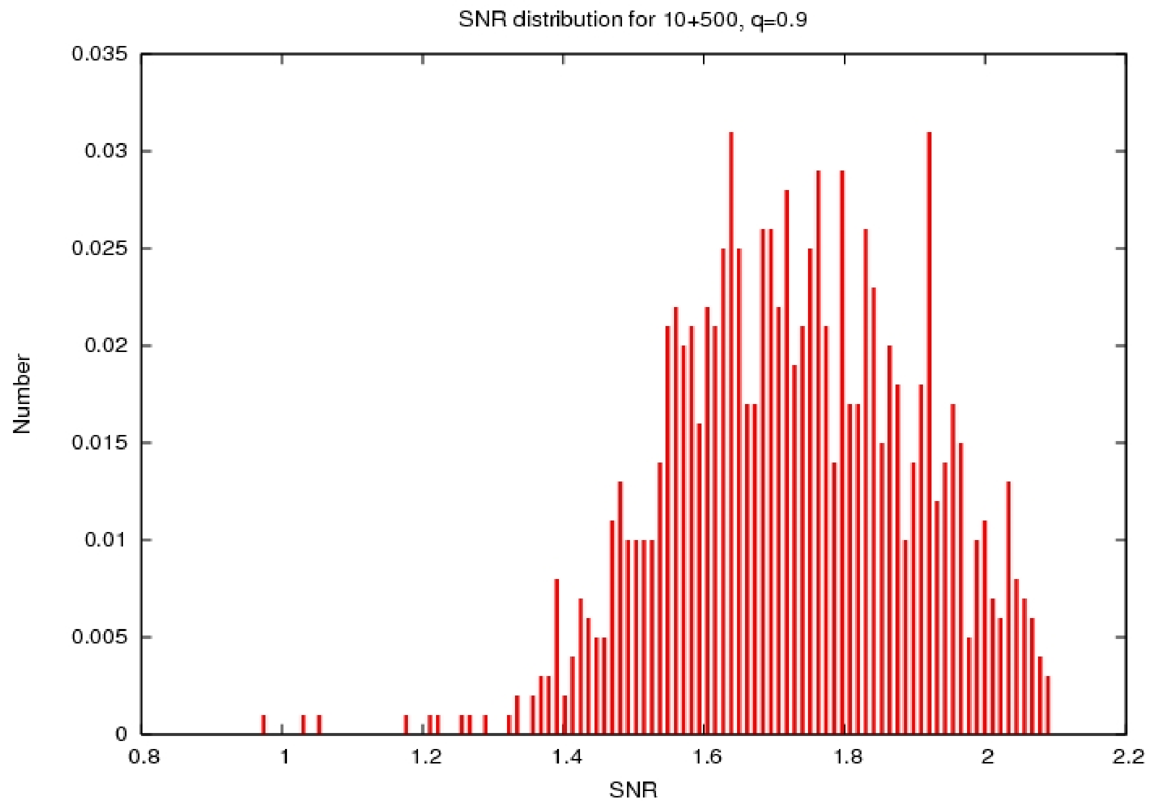
Model		Parameter									
		$\log(m)$	$\log(M)$	$\log(q)$	$\log(p_0)$	$\log(\phi_0)$	$\log(\theta_S)$	$\log(\phi_S)$	$\log(\theta_K)$	$\log(\phi_K)$	$\log(D)$
q=0.9	Mean	-1.37	-1.76	-1.45	-1.96	1.20	0.35	0.48	0.84	0.95	-1.19
	St. Dev.	0.255	0.211	0.197	0.392	0.692	0.276	0.347	0.677	0.724	0.292
	L. Qt.	-2.24	-1.90	-1.59	-2.04	0.71	0.18	0.26	0.31	0.38	-1.43
	Med.	-2.24	-1.76	-1.46	-1.92	0.90	0.36	0.45	0.66	0.79	-1.25
	U. Qt.	-2.24	-1.62	-1.32	-1.79	1.57	0.53	0.68	1.35	1.48	-0.99

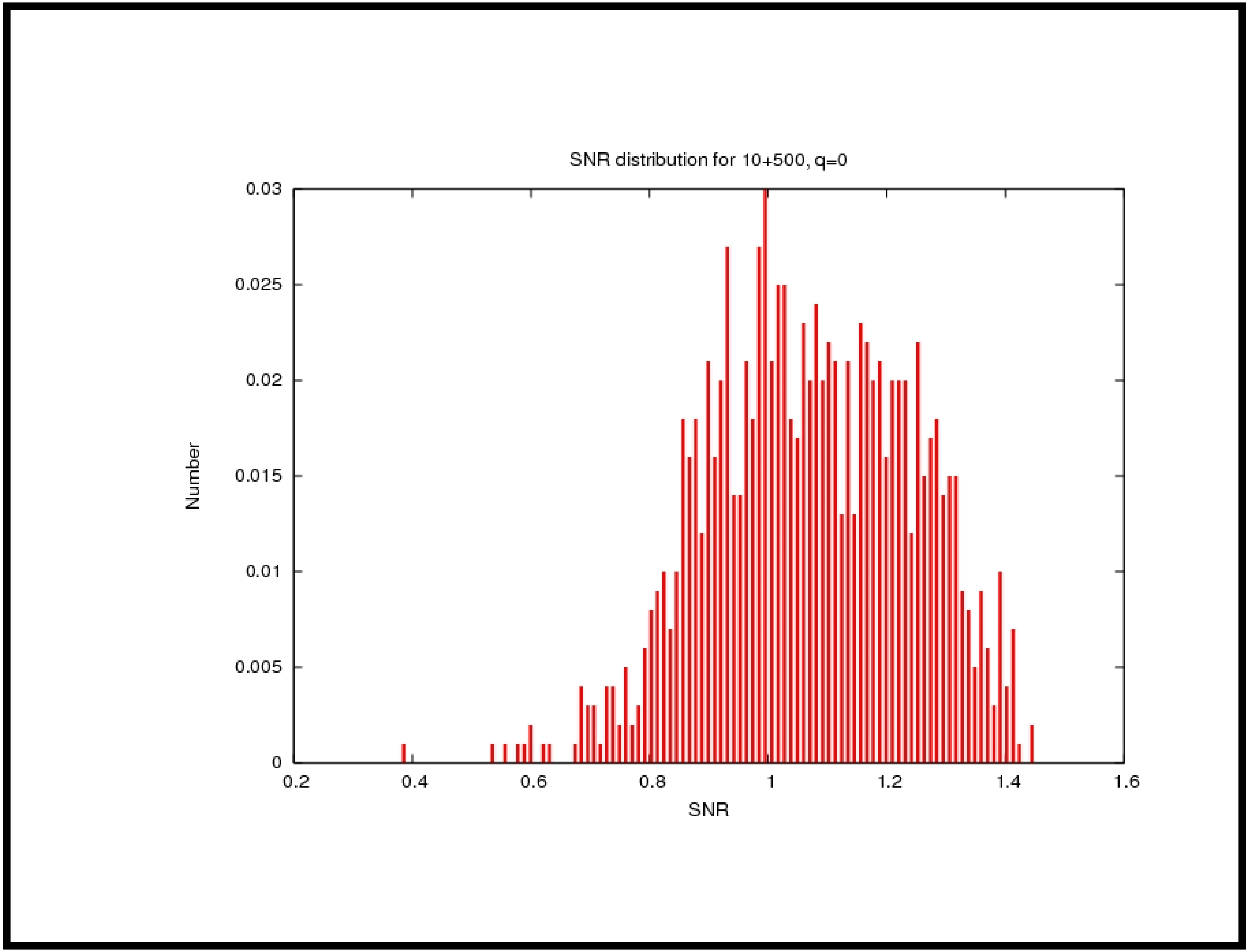
Table 1: Monte Carlo for the Fisher Matrix errors for black hole systems with $m = 10M_\odot$ and $q = 0.9$. Figures of the distribution of the logarithm to base ten of the ratio for each parameter.

Model		Parameter								
		$\log(m)$	$\log(M)$	$\log(p_0)$	$\log(\phi_0)$	$\log(\theta_S)$	$\log(\phi_S)$	$\log(\theta_K)$	$\log(\phi_K)$	$\log(D)$
q=0	Mean	-1.68	-2.05	-2.99	0.43	-0.30	-0.17	0.23	0.35	-0.48
	St. Dev.	0.191	0.187	0.555	0.837	0.271	0.346	0.749	0.765	0.349
	L. Qt.	-2.20	-2.18	-3.02	-0.19	-0.48	-0.41	-0.32	-0.22	-0.75
	Med.	-2.20	-2.05	-2.83	0.08	-0.30	-0.20	0.01	0.13	-0.57
	U. Qt.	-2.20	-1.91	-2.67	0.85	-0.11	0.05	0.68	0.77	-0.30

Table 2: Monte Carlo for the Fisher Matrix errors for black hole systems with $m = 10M_\odot$ and $q = 0$. Figures of the distribution of the logarithm to base ten of the ratio for each parameter.

☆ The SNR distribution for these two systems are the following


SNR distribution for 10+500, q=0.9
Number
SNR
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.8
1
1.2
1.4
1.6
1.8
2
2.2


SNR distribution for 10+500, q=0
Number
SNR
0.03
0.025
0.02
0.015
0.01
0.005
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

Summary

- ✰ We have built a complete waveform model that includes the inspiral, transition, plunge and ring–down phases for spinning binaries
- ✰ We have also run converging MC simulations for four different binary systems and have presented the statistics for the noise–induced FM errors and SNR distribution for two cases
- ✰ Run MC for an additional spin parameter $q = 0.3$